

# Are there really any $AdS_2$ branes in the euclidean (or not) $AdS_3$ ?

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## Abstract

We do not find any  $AdS_2$  branes, neither in the  $H_3^+$  nor the  $SL(2, \mathbb{R})$  WZNW model. We then reexamine the case of the branes that possess a  $su(2)$  symmetry: they would have to live on the boundary of  $AdS_3$ . This cannot be realized in an euclidean spacetime, but in the  $SL(2, \mathbb{R})$  WZNW model by analytical continuation.

## Introduction

The discussion of maximally symmetric branes in the euclidean (and lorentzian)  $AdS_3$  has received some attention recently [1–12]. It seems established that the possible maximally symmetric branes of the model possess either a  $sl(2, \mathbb{R})$  symmetry ( $AdS_2$  branes) or a  $su(2)$  symmetry (called spherical branes in [12]), according to the different possible gluing conditions for the currents on the boundary of the worldsheet. Using conformal field theory techniques already developped for Liouville field theory in [13, 14], the authors of [10–12] proposed a microcospic description of these branes. In [10] were proposed one point functions in the  $sl(2, \mathbb{R})$  and  $su(2)$  cases, which, as noticed in [12], turned out to be incorrect as the authors had wrong ansatz for its space time dependence. In [11], the one

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point function in the  $sl(2, \mathbb{R})$  case was proposed, in [12], the one point functions in the  $sl(2, \mathbb{R})$  (which coincides with the one of [11]), and  $su(2)$  cases as well as the boundary reflection amplitude were constructed. The Cardy condition was checked in both cases, leading to a continuous spectrum of boundary fields in the  $sl(2, \mathbb{R})$  case, and to a discrete and finite one in the  $su(2)$  case.

However, we believe that these results do not lie on solid grounds, as the functional relations satisfied by these one point functions and the boundary reflection amplitude were only partially solved (see the conclusion of [12]). We would like to emphasize that all functional relations *must* be solved. Thus, we do not consider the respective Cardy conditions as proven, as neither the quantities needed in the closed string channel (the one point function) nor in the open string channel (the boundary reflection amplitude in the  $sl(2, \mathbb{R})$  case) are perfectly under control. Some time ago, one of the authors of the present paper (B.P.) has emitted some doubts about the existence of these  $AdS_2$  branes, (see the conclusion of [15]), as it does not seem to be possible to construct coherently the boundary three point function (*i.e.* the scattering amplitude of open string states). So we decide to reexamine the problem once more. This paper is organized as follows: section one contains basic definitions and notations, as well as the relations taken from [12] that define the  $AdS_2$  and the  $su(2)$ -branes. In section two we check for the  $AdS_2$  branes the variational principle: it is satisfied without the need of adding any boundary action, so we do not find any source that renders the boundary problem interactive. It turns out also that it does not seem possible to construct a coherent conformal field theory in this case, so we propose to discard the existence of the  $AdS_2$  branes. In section three, we reexamine the validity of the factorization constraints in the  $su(2)$  case. They have no solution in an euclidean spacetime, but it might be possible to construct a consistent boundary conformal field theory in the  $SL(2, \mathbb{R})$  WZNW model (the results are those of [12], once the analytical continuation from euclidean to lorentzian of the spacetime and worldsheet coordinates is performed). A striking similarity with the boundary conditions that appear in Liouville field theory on the euclidean  $AdS_2$  considered by Zamolodchikov and Zamolodchikov in [14] is discussed.

## 1 Preliminaries [12]

The symmetric space  $H_3^+$  consists of hermitian  $2 \times 2$  matrices  $h$  with determinant  $\det h = 1$  and positive trace. We parametrize this space through coordinates  $(\phi, \gamma, \bar{\gamma})$  such that

$$h = \begin{pmatrix} e^\phi & e^\phi \bar{\gamma} \\ e^\phi \gamma & e^\phi \gamma \bar{\gamma} + e^{-\phi} \end{pmatrix} . \quad (1)$$

$\phi$  is real and  $\gamma$  is a complex coordinate with conjugate  $\bar{\gamma}$ . The space  $H_3^+$  is equipped with the following metric and  $H$ -field,

$$ds^2 = d\phi^2 + e^{2\phi} d\gamma d\bar{\gamma} \quad , \quad H = 2 e^{2\phi} d\phi \wedge d\bar{\gamma} \wedge d\gamma \quad . \quad (2)$$

We shall consider the following 2-form potential  $B'$  for  $H$ :

$$B' = e^{2\phi} d\gamma \wedge d\bar{\gamma} \quad .$$

The B-field is imaginary so the theory is non-unitary. The action functional for closed strings moving on  $H_3^+$  then reads<sup>3</sup>:

$$S(\phi, \gamma, \bar{\gamma}) = \frac{k}{\pi} \int dz d\bar{z} (\partial\phi\bar{\partial}\phi + e^{2\phi}\bar{\partial}\gamma\partial\bar{\gamma}) \quad . \quad (3)$$

*The currents.* Let us introduce the following matrices

$$T_+ = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}, \quad T_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_0 = \frac{1}{2} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (4)$$

These are matrix representatives of the Lie algebra  $sl(2, \mathbb{R})$ , i.e. they obey the relations  $[T_0, T_{\pm}] = \pm T_{\pm}$  and  $[T_-, T_+] = 2T_0$ . For the chiral currents we use

$$J(\bar{z}) := k h^{-1} \bar{\partial} h \quad \bar{J}(z) := -k \partial h h^{-1}.$$

When we expand them according to  $J(\bar{z}) = T_+ J^+ + T_- J^- + 2T_0 J^0$ , we obtain expressions for the components

$$J^-(\bar{z}) := k e^{2\phi} \bar{\partial} \gamma \quad (5)$$

$$J^0(\bar{z}) := k (\bar{\partial}\phi - e^{2\phi} \bar{\gamma} \bar{\partial}\gamma) \quad (6)$$

$$J^+(\bar{z}) := k (\bar{\gamma}^2 e^{2\phi} \bar{\partial}\gamma - \bar{\partial}\bar{\gamma} - 2\bar{\gamma} \bar{\partial}\phi) \quad . \quad (7)$$

The components of the anti-holomorphic currents are constructed in an analogous way. Both sets of currents are related by complex conjugation  $(J^{\pm})^* = (\bar{J})^{\mp}$  and  $(J^0)^* = -\bar{J}^0$ .

*The  $AdS_2$  branes.* They correspond to surfaces which are characterized by the equation

$$tr \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} h = c \quad .$$

where  $c$  is a constant. In terms of the coordinates introduced above one gets the equation

$$e^{\phi} (\gamma + \bar{\gamma}) = c \quad . \quad (8)$$

The currents satisfy the following relations on the boundary of the worldsheet<sup>4</sup>

$$J^{\pm}(\bar{z}) = -\bar{J}^{\mp}(z) \quad , \quad J^0(\bar{z}) = -\bar{J}^0(z) \quad . \quad (9)$$

This implies that the current obeys  $(J^{\pm})^* = -J^{\pm}$  and  $(J^0)^* = J^0$  at  $z = \bar{z}$ .

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<sup>3</sup>We correct here a misprint of [12].

<sup>4</sup>Same as above.

*Branes that preserve a  $su(2)$  symmetry.* They are such that

$$\text{tr} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} h = c .$$

with  $c$  constant. This equation can be rewritten as:

$$e^\phi(\gamma\bar{\gamma} + 1) + e^{-\phi} = c. \quad (10)$$

The currents satisfy<sup>5</sup> along the boundary  $z = \bar{z} : J^\pm = \bar{J}^\pm, J^0 = \bar{J}^0$ . So we have  $(J^\pm)^* = J^\mp, (J^0)^* = -J^0$ , *i.e.* a  $su(2)$  current algebra on the boundary of the worldsheet.

## 2 $AdS_2$ branes

The star conditions for the currents give the following boundary conditions for the fields at  $z = \bar{z}$ :

$$(\partial - \bar{\partial})\phi = -ce^\phi\bar{\partial}\gamma, \quad (11)$$

$$\gamma + \bar{\gamma} = ce^{-\phi}, \quad (12)$$

$$\partial\bar{\gamma} + \bar{\partial}\gamma = 0. \quad (13)$$

If one sets:  $z = \tau + i\sigma, \bar{z} = \tau - i\sigma, \partial_z = \frac{1}{2}(\partial_\tau - i\partial_\sigma), \partial_{\bar{z}} = \frac{1}{2}(\partial_\tau + i\partial_\sigma)$ , then one can rewrite the boundary conditions as

$$i\partial_\sigma\phi = \frac{c}{2}e^\phi(\partial_\tau + i\partial_\sigma)\gamma, \quad (14)$$

$$\gamma + \bar{\gamma} = ce^{-\phi}, \quad (15)$$

$$i\partial_\sigma(\gamma - \bar{\gamma}) = -c\partial_\tau e^{-\phi}. \quad (16)$$

Then, using these conditions, the variational principle states that at  $\sigma = 0$ ,

$$\delta\phi\partial_\sigma\phi + \frac{1}{2}e^{2\phi}(\delta\gamma(\partial_\sigma + i\partial_\tau)\bar{\gamma} + (\partial_\sigma - i\partial_\tau)\gamma\delta\bar{\gamma}) = 0. \quad (17)$$

The variational principle is thus satisfied *without the need of adding any boundary term in the action*. There is no potential on the boundary that provides the dynamical source for the existence of the  $AdS_2$  branes. Moreover, a straightforward computation shows that the one point function proposed in [11,12] (a boundary interaction is implied there) *does not* satisfy the factorization constraint arising when one considers the degenerate field with spin  $1/2b^2$  (this even in the simplest case of the straight brane). Let us give another argument: if one considers a four point function of boundary operators, then it contains a singularity at  $z = x$  (see [15]), where  $z$  is the worldsheet variable and  $x$  the space-time coordinate. On the boundary,  $x$  and  $z$  are real variables. Unlike the bulk case, such a singularity on the boundary is not acceptable. We are not able to construct a coherent conformal field theory in this case.

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<sup>5</sup>Same as above.

### 3 $su(2)$ -branes

Let us remind that to construct the one point function in this case, one first starts with a bulk two point function, where one of the fields has a spin  $1/2$ , the other a spin  $j$  (see [14] for such a discussion). There are two equivalent ways of expressing this two point function: either one first merges the two operators in the bulk (this is given by a special case of the three point function structure constant), then one approaches the resulting field to the boundary, which is given by the one point function; in the other channel, the two fields in the bulk do approach the boundary, which gives a product of two one point functions  $U_\rho(1/2)U_\rho(j)$  ( $\rho$  is the boundary label). It is not clear whether the validity condition for such a factorization has been properly discussed in [10, 12]: one requires a two point function to factorize into a product of two one-point function, but, as mentioned in [14], this can be the case only when the fields are very far apart in the *spacetime* when they approach the boundary; to have the geodesical distance between the fields become infinite as they approach the boundary of the worldsheet, the fields should be at the boundary of  $H_3^+$ , *i.e.* at  $\phi = +\infty$ , and consequently  $\gamma\bar{\gamma} + 1 = 0$  for the relation written in (10) to be satisfied<sup>6</sup>. This equation does not have any solution in the euclidean spacetime, so it seems that these  $su(2)$ -branes do not exist in  $H_3^+$ <sup>7</sup>. However, there would be a solution in the  $SL(2, \mathbb{R})$  model, where  $\gamma, \bar{\gamma}$  are substituted by real fields  $a, b$ <sup>8</sup>. In this case, the boundary conformal field theory constructed in [12] seems to be coherent (it is straightforward to check that the one point function given in the equation (3.41) of this paper indeed satisfies *all* the factorization constraints - only one factorization constraint involving the degenerate field with spin  $j = 1/2$  was solved in [12]). We would like to point out that the relation

$$e^\phi \sim \frac{c}{1 + ab},$$

valid at  $1 + ab = 0$ ,  $\phi = +\infty$ , is very reminiscent of what was found for Liouville field theory on the pseudosphere (euclidean  $AdS_2$ ) [14]. In this case the metric on the Lobachevski plane is

$$ds^2 = e^{\phi_L(z, \bar{z})} |dz|^2$$

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<sup>6</sup>We take here the opportunity to mention that once again, the boundary conditions for the fields make the variational principle satisfied *without the need of extra boundary term in the action*.

<sup>7</sup>We understand that this discussion is somewhat speculative and should be handled with care: the requirement that the two point function decays into a product of one point functions when the fields are taken very far apart is very formal here, as it holds in principle in *unitary* quantum field theories. In the case of Liouville field theory on the pseudosphere [14], it can be used *a priori* as this theory is believed to be a unitary conformal field theory, but it turns out *a posteriori* that the correlation functions in the bulk are divergent, which is certainly not an expected feature for a unitary conformal field theory. One could object the use of such an argument in the  $H_3^+$  model, which is known to be a *non unitary* model. However, it might be that it is the non unitarity of the model that prevents the construction of any D-branes.

<sup>8</sup>The worldsheet also becomes lorentzian.

where  $\phi_L$  is the Liouville field, and

$$e^{\phi_L(z,\bar{z})} = \frac{4R^2}{(1 - z\bar{z})^2},$$

$R$  is interpreted here as the radius of the pseudosphere. It was shown in [14] that the boundary fields live on the boundary of the surface parametrized by  $z\bar{z} = 1$  (called the absolute), and that the possible boundary conditions are in one to one correspondence with the degenerate representations of the Virasoro algebra. We believe it is no accident if the boundary three point function of  $SL(2, \mathbb{R})$  model can be written in terms of the boundary three point function in Liouville field theory <sup>9</sup> (we discard on both sides the worldsheet and space-time dependence, and consider only the structure constants).

$$C_{j_3, j_2, j_1}^{\rho_3, \rho_2, \rho_1} = \frac{\Gamma(2 + b^{-2} + j_3 + j_2 + j_1)}{\Gamma(2 + b^{-2} + j_3 + j_2 + j_1 - m)} D_{-bj_3, -bj_2, -bj_1}^{-b\rho_3, -b\rho_2, -b\rho_1}$$

where  $C$  stands for the boundary three point function in the  $SL(2, \mathbb{R})$  WZNW model,  $D$  is the boundary three point function in Liouville field theory; a boundary operator is labelled by its spin  $j$ , and its left and right boundary conditions  $\rho_1$  and  $\rho_2$ . The labels here are integers submitted to the conditions <sup>10</sup>

$$\rho_2 = \rho_1 + j_1 - n, \quad j_3 = j_2 + j_1 - m, \quad n, m \in \mathbb{N}$$

We do not infer that these two models are equivalent: it was shown in [14] in the pseudosphere case, the possible boundary conditions are parametrized by two positive integers  $(s, t)$ , and if  $s > 1$ , the one point function does not have any usual classical limit. In the WZNW model considered here, the boundary conditions are parametrized by one positive integer only, and the one point function does have a smooth classical limit. The situation is very analogous to Liouville theory on the pseudosphere with  $s = 1$ .

## Acknowledgments

B.P. would like to thank S. Theisen and Al.B. Zamolodchikov for discussions, and in particular N. Drukker for his interest. Work supported by EU under contract HPRN-CT-2000-00122.

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<sup>9</sup>The boundary three point function is constructed following the lines of [16]: the normalization of the boundary operators can be found in [12], equ. 4.40, and the fusion matrix is given by equ. 26 of [15] (the parameter  $-b^{-2}$  in this formula should be substituted by  $b^{-2}$ , and the superscript  $MM$  that stands for minimal model should be replaced by LFT for Liouville field theory.

<sup>10</sup>These conditions make the four point function of boundary operators regular at  $z = x$ .

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